Bezrukavnikov's equivalence

$G/_k$ split algebraic group. $G^\vee$ its Langlands dual.

X characters $\supset R$ weight lattice, $X^\vee$ cochar $\supset R^\vee$

W_f finite Weyl group $W = X^\vee \rtimes W_f$ affine Weyl group

$F = k((t))$, $\mathcal{O} = k[[t]]$, G_F, $G_\mathcal{O}$ groups

Fix a Borel B and Iwahori $I = ev^{-1}(B)$.

H affine Hecke algebra $= C_c(I \backslash G_F / I)$

$= \bigoplus_w \mathbb{Z}[v^{\pm 1}] H_w$, $\quad H_{ww'} = H_w H_{w'}$, if $l(ww') = l(w) + l(w')$

$(H_s + v)(H_s - v^{-1}) = 0$, s simple reflection.

$H \supset H_f = \bigoplus_{w \in W_f} \mathbb{Z}[v^{\pm 1}] H_w$ finite Hecke algebra.

Also the lattice part θ_λ, $\lambda \in X^\vee$, such that

$\theta_\lambda \theta_\mu = \theta_{\lambda+\mu}$. $\theta_\lambda = H_\lambda$ if $\lambda \in X^\vee_+$.

$\mathcal{L} = \bigoplus_{\lambda \in X^\vee} \theta_\lambda$. $\quad H^{sph} = Z(H) = \mathcal{L}^{W_f} \simeq C_c(G_\mathcal{O} \backslash G_F / G_\mathcal{O})$.

Relation:

$$H_\alpha H_{s_\alpha \lambda} - H_\lambda H_{s_\alpha} = (v - v^{-1}) \frac{H_\lambda - H_{s_\alpha \lambda}}{1 - H_\alpha}$$

Example: $G = GL_2$, $X = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^*$, $X^\vee = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$.

simple coroot $\alpha = e_1 - e_2$. $W_f = \mathfrak{S}_2 = \{1, s\}$, $s: e_1 \leftrightarrow e_2$.

for $\lambda \in X^\vee$, denote t_λ the translation of λ acting on $X^\vee$.

$\varpi = t_{e_1} s$, then $l(\varpi) = 0$, ϖ is the generator of length 0 ele.

$s_0 = t_\alpha s$ also a reflection. $W = \langle t_{e_1}, t_{e_2}, s \rangle = \langle \varpi, s, s_0 \rangle$.

$\varpi s = s_0 \varpi$, $\varpi s_0 = s \varpi$, $\Rightarrow \varpi^2 \in Z(W)$. $\varpi^2 = t_{e_1+e_2}$

$\theta_{e_1} = H_{e_1} = H_\varpi H_s$, $\theta_{e_2}^{-1} = \theta_{-e_2} = H_\varpi^{-1} H_s \Rightarrow \theta_{e_2} = H_s^{-1} H_\varpi = H_\varpi H_{s_0}^{-1}$.

$\Rightarrow Z(H) = \mathbb{Z}[v^{\pm 1}][\theta_{e_1}, \theta_{e_2}]^{\mathfrak{S}_2} = \mathbb{Z}[v^{\pm 1}][\theta_{e_1} + \theta_{e_2}, \theta_{e_1}\theta_{e_2} = H_\varpi^2]$.

While $H^{sph}_\mathbb{C} = K_0(\mathrm{Rep}\, GL_2)_\mathbb{C}$ is generated by $\mathbb{C}^2$ and det

natural rep

On constructible side: a possible categorification: $D^b_I(\mathcal{F}l)$

On coherent side: $\mathfrak{g}^\vee$ the Lie algebra of $G^\vee$.

$\mathcal{B}$ = flag variety $= \{\mathfrak{b} \subset \mathfrak{g}^\vee \text{ Borel}\}$

$\mathcal{N} \subset \mathfrak{g}^\vee$ the nilpotent cone,

$\tilde{\mathcal{N}} = T^*\mathcal{B} = \{(\mathfrak{b}, x) : x \in \mathrm{nil}(\mathfrak{b})\} \longrightarrow \mathcal{N}$ Springer resolution.

$St = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ the Steinberg variety $= \{(\mathfrak{b}, \mathfrak{b}', x) : x \in \mathrm{nil}(\mathfrak{b}) \cap \mathrm{nil}(\mathfrak{b}')\}$.

$G \curvearrowright \mathcal{B} \times \mathcal{B}$ by diagonal, with orbits

$O_w = \{(\mathfrak{b}, \mathfrak{b}') : \text{relative position } w\}$, $w \in W_f$.

$St = \bigsqcup_{w \in W_f} T^*_{O_w}(\mathcal{B} \times \mathcal{B})$. tangent to G-orbit is just $\mathrm{nil}(\mathfrak{b}) \cap \mathrm{nil}(\mathfrak{b}')$

$\tilde{\mathcal{N}}$ has $G^\vee \times \mathbb{G}_m$ action: $G^\vee \curvearrowright \mathcal{B}$, $\mathbb{C}^\times \curvearrowright$ fiber

$\Rightarrow$ St has $G^\vee \times \mathbb{G}_m$ action: $(g, z)(\mathfrak{b}, \mathfrak{b}', x) = (g \cdot \mathfrak{b}, g \cdot \mathfrak{b}', z^2 g x)$.

Then $K^{G^\vee \times \mathbb{G}_m}(St)$, the K-group of coherent sheaves, has convolution product, isomorphic to H (Kazhdan-Lusztig)

Anti-spherical module:

$K^{G^\vee \times \mathbb{G}_m}(St) \curvearrowright K^{G^\vee \times \mathbb{G}_m}(\tilde{\mathcal{N}})$ by convolution

$$M_{asph} = K^{G^\vee \times \mathbb{G}_m}(\tilde{\mathcal{N}}) = K^{G^\vee \times \mathbb{G}_m}(\mathcal{B}) = K^{B^\vee \times \mathbb{G}_m}(pt) = \mathbb{Z}[v^{\pm 1}][X^\vee].$$

Also $sgn: H_f \to \mathbb{Z}[v^{\pm 1}]$, $H_s \mapsto -v$ $\quad \Rightarrow M_{asph} = H \otimes_{H_s} sgn$

Pf (Kazhdan-Lusztig): show the two actions are same and faithful.

A possible categorification $D^b(Coh(St))$. ✗

In fact: Bezrukavnikov:

$$D^b_I(\mathcal{Fl}) \cong DG\,Coh(\tilde{\mathcal{N}} \overset{L}{\times}_{\mathcal{N}} \tilde{\mathcal{N}})$$

(Arkhipov-Bezrukavnikov: $D^b Coh(\tilde{\mathcal{N}}) \cong D^I_{IW}$)

Iwahori-Whittaker category

Langlands motivation

K (number) field Goal: $G_K = \mathrm{Gal}(\overline{K}/K)$

- $K (= \mathbb{F}_p)$ finite field, $G_K = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$.
- $K (= \mathbb{Q}_p)$ local field valuation: $v: K \to \mathbb{Q}$.

  ring of integers: $\mathcal{O}_K$, uniformizer: ϖ.

$$0 \to I_{L/K} \to \mathrm{Gal}(L/K) \to \mathrm{Gal}\big((\mathcal{O}_L/(\varpi_L)) / (\mathcal{O}_K/(\varpi))\big) \to 1$$

$I_{L/K}$ = Inertial group, $\mathrm{Gal}\big((\mathcal{O}_L/(\varpi_L)) / (\mathcal{O}_K/(\varpi))\big) = \mathbb{Z}/n\mathbb{Z}$

I has a filtration

$$I = I_0 \supset I_i = \{\sigma \in I : \sigma(\varpi)\varpi^{-1} \in 1 + (\varpi_L)^i\}$$

$I_0/I_1 \subseteq (\mathcal{O}_L/(\varpi_L))^\times$ order prime to p, $\Rightarrow$ tamely

$I_i/I_{i+1} \subseteq (1+(\varpi_L)^i)/(1+\varpi_L)^{i+1}$, p-group, $\Rightarrow$ wild

Similarly $0 \to I \to \mathrm{Gal}(\overline{K}/K) \to \hat{\mathbb{Z}} \to 0$

$\Rightarrow$

K^{sep}

| Gal: pro-p group

K^{tame}

| Gal $= \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$ ← note the Frobenius action

K^{ur}

| Gal $= \hat{\mathbb{Z}}$

K

Local class field theory

Weil group

$$\begin{tikzcd} & & W_K \arrow[r, two heads, "\chi"] \arrow[d, hook] & \mathbb{Z} \arrow[d] & \\ 0 \arrow[r] & I \arrow[r] & G_K \arrow[r, two heads] & \hat{\mathbb{Z}} \arrow[r] & 0 \end{tikzcd}$$

Artin map: $\widehat{K^\times} \xrightarrow{\sim} G_K^{ab} \Rightarrow K^\times \longrightarrow W_K^{ab}$.

$\Rightarrow$ local Langlands for $n=1$:

$$\{\text{Irr rep of } GL_1(K)\} \xleftrightarrow{1:1} \{\text{1-dim rep of } W_K\}$$

General: tamely unramified

$$\{\text{Irr sm rep of } G(K)\} \overset{1:1}{\longleftrightarrow} \{\text{WD rep of } W_K \text{ Frob-s.s.}\}$$

WD rep: (r, N): $r: W_K \to G^\vee(\mathbb{C})$, $N \in \mathcal{N} \subset \mathfrak{g}^\vee$

s.t. $Ad_{r(g)} N = q^{-\chi(g)} N$.

Relation: rep of $H = C_c(I\backslash G_F/I) = e_I C_c(G_F) e_I$

$\overset{1:1}{\longleftrightarrow}$ rep of G_F with non-zero I-fixed point.

rep of $K^{G^\vee \times \mathbb{G}_m}(St) \Leftrightarrow K^{G^\vee \times \mathbb{G}_m}$(Springer fiber)

& $Z_{G^\vee \times \mathbb{G}_m}(e)$ looks like WD-rep.

If require G_0-stable $\Rightarrow$ Satake isomorphism

Global class field theory

F number field, $\mathbb{A}$ = adéles of F

$$\begin{array}{lccc} \text{Art}: & F^\times\backslash\mathbb{A}^\times/(F_\mathbb{R}^\times)^\circ & \xrightarrow{\sim} & G_F^{ab}. \\ \text{for any place } v & \uparrow & \circlearrowleft & \uparrow \\ \text{Art}: & F_v^\times & \longrightarrow & G_{F_v}^\times \end{array}$$

Similarly global Langlands

$$\{\text{some rep of } G(\mathbb{A})\} \longleftrightarrow \left\{\begin{matrix}\text{some Galois rep} \\ G_F \to G^\vee(\mathbb{C})\end{matrix}\right\}$$

Also for function field:

$$\left\{\begin{matrix}\text{cuspidal smooth} \\ \text{automorphic rep of } G(\mathbb{A})\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\text{continuous Galois rep} \\ G_F \to G^\vee(\mathbb{C})\end{matrix}\right\}$$

Pf (Lafforgue: $\rightarrow$)

Shtuka $\Rightarrow$ the trivial one $= C_c([G])$.

some action from ev, coev $\Rightarrow$ Hecke operator on $C_c([G])$.

Hecke finite $\Rightarrow$ cuspidal rep.

pseudo-representation $\Rightarrow$ representations.